

Introduction of Stress Wave Propagation

Effect of a sharply applied localized disturbance in a medium soon transmits or spreads to other part of medium. The local excitation of medium is not instantaneously detected at position that is at a distance from the region of excitation, it takes time for a disturbance to propagate from its source to other position. e.g. an earthquake or an underground nuclear explosion is recorded in another continent well after it has occurred.

Stress waves originate in the forced motion of a portion of a deformable medium, as elements of the medium are deformed the disturbance is transmitted from one point to the next and the disturbance, or stress wave, progresses through the medium.

In this process the resistance offered to deformation by the consistency of medium, as well as the resistance to motion offered by inertia, must be overcome. Deformability and inertia are essential properties of a medium for the transmission of wave motion.

If the medium were not deformable, any part of the medium would immediately experience a disturbance in the form of an internal force or acceleration upon application of a localized excitation.

If a hypothetical medium were without inertia, there would be no delay in the displacement of particles and the transmission of the disturbance from particle to particle would be effected instantaneously to the most distant particle.

But real materials are of course deformable and possess mass and thus all real materials transmit stress wave. Propagation of stress wave may be through liquid, solid medium or air.

The finite velocity of wave in fluid can be expressed in $(k/\rho)^{1/2}$, ρ : density, k : bulk modulus, fluid cannot sustain finite shear stresses, the dilatational wave is the only type of wave motion that can be propagated through it.

In our discussion, it is focused on the propagation of waves in linear elastic continuum solids, so that properties such as density or elastic constants are considered to continuous function.

In elastic solids, two type of elastic wave may be propagated

1. solid will transmit tensile and compressive stresses (P wave, longitudinal wave), the motion of particles will be in the direction of the wave motion, the speed of propagation is $c_d^2 = (\lambda + 2\mu)/\rho$, λ , μ : Lamé' constants

2. solid may transmit shear stress and the motion of particles in transverse to the direction of propagation, the speed of propagation is

$$c_s^2 = \mu/\rho$$

SH wave and SV wave represent the horizontally and vertically polarized shear waves.

Consider a solid body subjected to an external disturbance $F(t)$ applied at a point P at $t = 0$, if the greatest velocity of propagation is c , the disturbed region is bounded by sphere centered at P with radius ct .

A wavefront will be associated with the outward spreading disturbance, particles ahead of the front will have no motion.

When a solid medium is deformed, both distortional (shear) and dilatational (longitudinal) waves will normally be produced.

When a wave of either type impinges on a boundary or interface, wave of both types may be generated by reflection, refraction or diffraction.

Elastic waves may be propagated along the surface of a solid, there are known as surface wave (Rayleigh wave), and may be propagated along near the region of interface of two medium, there are known as intersurface wave (Stonely wave).

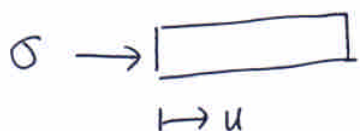
Shock wave may be formed in medium when the velocity of propagation of large disturbance in it is greater than that of smaller ones.

equation of motion

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i$$

f_i : body force ρ : density

for one dimension case with $f_i = 0$



$$\sigma_{11} = \sigma \quad \epsilon_{11} = \epsilon$$
$$u_{,1} = u$$

$$\sigma_{11,1} = \rho \ddot{u}_{,1}$$

$$\sigma_{11} = \sigma = E \cdot \epsilon_{11} = E u_{,1,1}$$

$$\therefore E u_{,1,1} = \rho \ddot{u}$$

$$\text{or} \quad E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

wave equation.
(hyperbolic type P.D.E)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2}$$

$$c_0 = \sqrt{\frac{E}{\rho}}$$

c_0 : wave speed.

$$\text{let} \quad \xi = x - c_0 t$$

$$\eta = x + c_0 t$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$\frac{\partial u}{\partial t} = -c_0 \frac{\partial u}{\partial \xi} + c_0 \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial \xi^2} + c_0^2 \frac{\partial^2 u}{\partial \eta^2} - 2 c_0^2 \frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} = 0$$

$$\Rightarrow 4 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\text{i.e. } u(\xi, \eta) = f(\xi) + g(\eta)$$

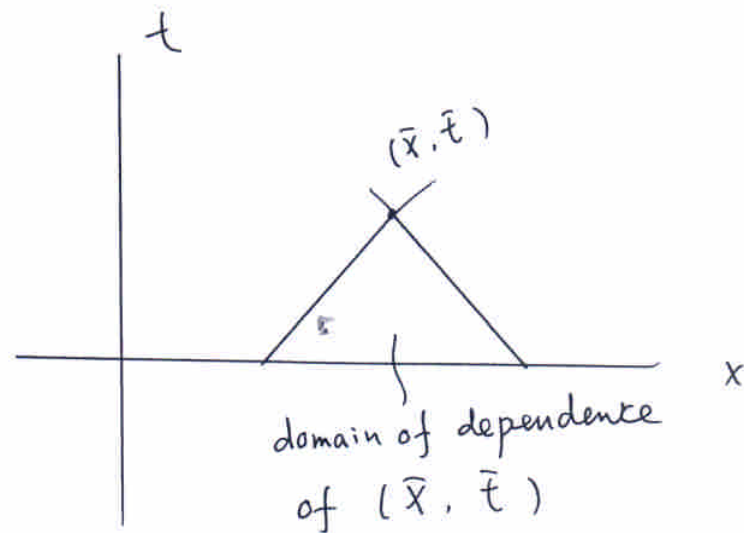
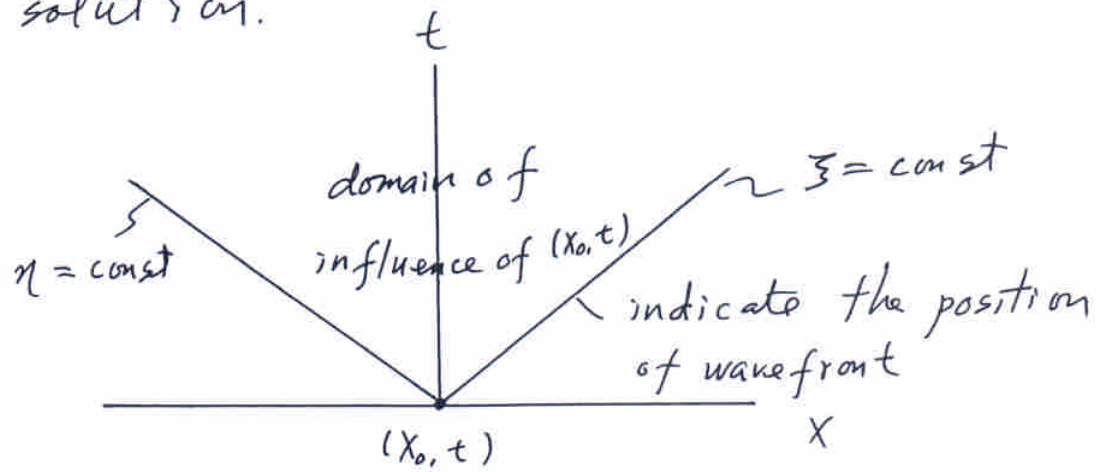
$$\text{or } u(x, t) = f(x - c_0 t) + g(x + c_0 t)$$

this is the D'Alembert solution for wave equation.

so that longitudinal waves propagation at the velocity c_0 in the rod.

for $\xi = x - c_0 t = \text{const}$
 $\eta = x + c_0 t = \text{const}$ } represents a straight line in $x-t$ plane

these lines are called characteristics of the solution.



for $f(x - ct)$, if $x - ct = \text{const}$.

i.e. increasing time requires increasing value of x

to maintain the argument of the function

or wave propagating in $+x$ direction

$g(x + ct)$: wave propagating in $-x$ direction

Example. For simplicity of illustration, consider only displacement initial conditions. Thus, suppose

$$y(x, 0) = U(x) = \begin{cases} 1, & -a < x < a \\ 0, & |x| > a \end{cases} \quad (1.1.34)$$

$$\dot{y}(x, 0) = V(x) = 0.$$

Then

$$y(x, t) = \frac{1}{2} \{U(x - c_0 t) + U(x + c_0 t)\}. \quad (1.1.35)$$

The behaviour of the string at different instants of time is shown in Fig. 1.5.

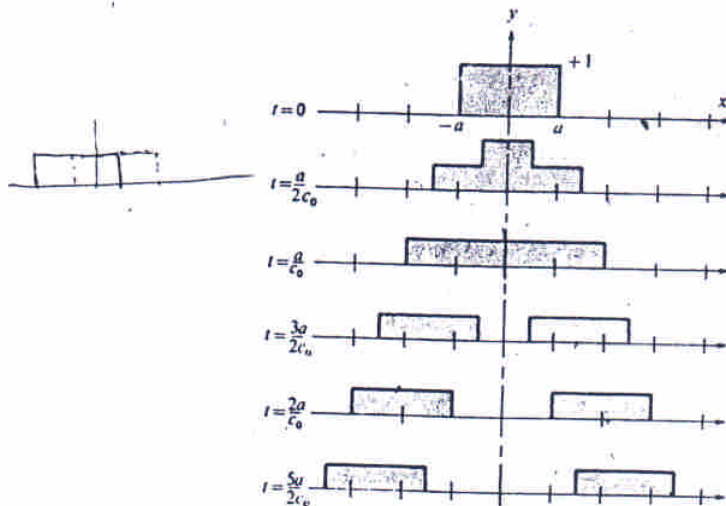


FIG. 1.5. Propagation of an initial condition displacement in a string.

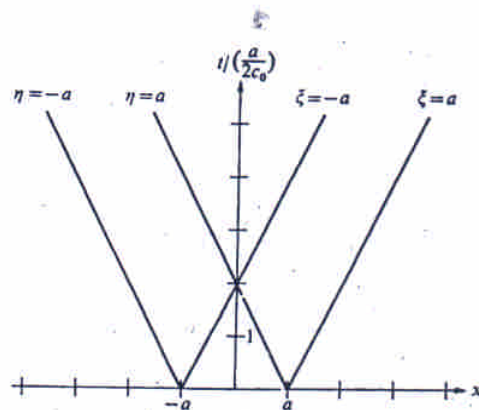
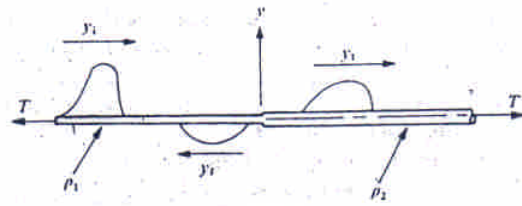


FIG. 1.6. Characteristic plane representation of the propagation of a disturbance in a string

Reflection and transmission at discontinuities



$$u_i = f_1(x - c_1 t) \quad u_t = f_2(x - c_2 t)$$

$$u_r = g_1(x + c_1 t)$$

$$c_1 = \sqrt{\frac{E_1}{\rho_1}}$$

$$c_2 = \sqrt{\frac{E_2}{\rho_2}}$$

at discontinuity, velocity, & strain component.
be continuous

$$\left. \begin{aligned} u_i + u_r &= u_t \\ \dot{u}_i + \dot{u}_r &= \dot{u}_t \\ \frac{\partial u_i}{\partial x} + \frac{\partial u_r}{\partial x} &= \frac{\partial u_t}{\partial x} \end{aligned} \right\} \text{ at } x=0$$

$$\dot{u}_i = -c_1 f_1' \quad \dot{u}_r = c_1 g_1' \quad \dot{u}_t = -c_2 f_2'$$

$$\therefore -c_1 (f_1' - g_1') = -c_2 f_2'$$

$$f_1' + g_1' = f_2' \quad \left[\frac{\partial u_i}{\partial x} = f_1', \frac{\partial u_r}{\partial x} = g_1', \frac{\partial u_t}{\partial x} = f_2' \right]$$

$$\therefore f_2' = \frac{2c_1}{c_1 + c_2} f_1' \quad g_1' = \frac{c_1 - c_2}{c_1 + c_2} f_1'$$

$$\text{if } c_1 = c_2$$

$$f_2' = f_1', \quad g_1' = 0$$

$$\text{i.e. } u_t = u_i \quad \text{no reflection}$$

$$\text{if } c_2 \gg c_1 \quad (\text{such as fixed end})$$

$$f_2' = 0 \quad g_1' = -f_1' \quad \text{totally reflected.}$$

2.2.1. Reflection from fixed and free ends

Boundary conditions for a semi-infinite rod analogous to those for a string (see Table 1.1) can be formulated. The image method can be applied to the two simplest cases of end termination—the free and fixed ends. Again, the string results can be directly applied. However, there are one or two aspects of the phenomena in rods that deserve further emphasis. Consider first an incident pulse on a fixed boundary. The boundary condition is given by

$$u(0, t) = 0. \quad (2.2.1)$$

An image displacement pulse system is set up that will satisfy these boundary conditions and is shown in Fig. 2.4(a). Also shown is the corresponding stress

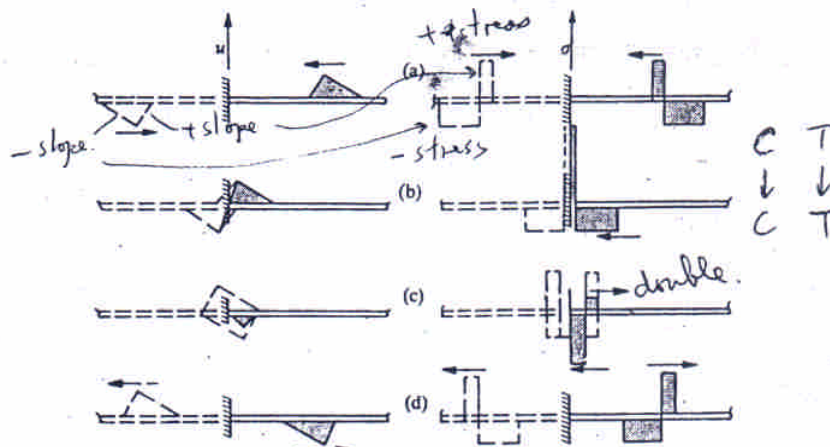


FIG. 2.4. Image displacement and stress pulse propagation, interaction, and reflection from a fixed boundary.

pulse situation. During steps (b) and (c) in the figure, interaction at the boundary occurs. The main point of this study is brought out in the stress-wave interaction in step (b). It is seen that the stresses superimpose to give double peak value. This is also occurring in step (c), where double the value

of the compressive part is evident. This stress multiplication phenomenon is characteristic of the fixed boundary. Proceeding to part (d), after interaction has occurred, it is seen that the reflected stress pulse is identical to the incident pulse. Thus, compression has reflected as compression and tension as tension. This also is a characteristic of the fixed boundary.

Now consider a free-end boundary condition, as given by

$$\partial u(0, t)/\partial x = 0. \quad \epsilon = 0, \quad \sigma = 0 \quad (2.2.2)$$

Again apply the image method to the reflection problem. The appropriate image system is shown in Fig. 2.5(a). As is evident in the interaction steps (b)

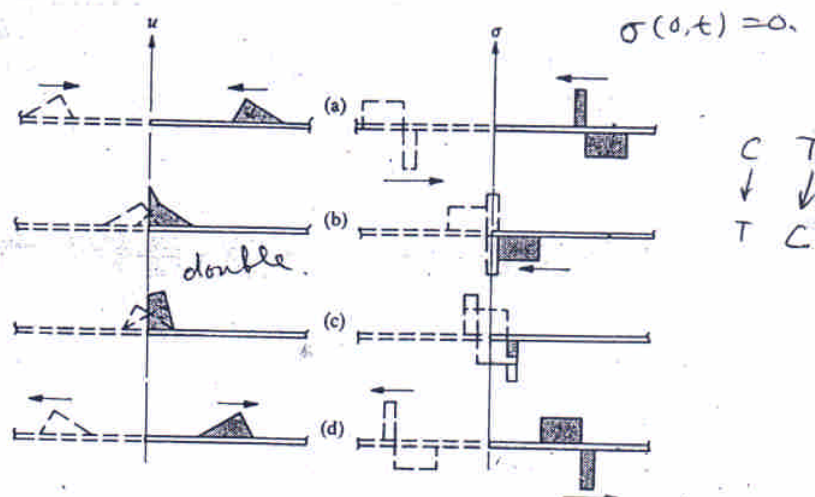
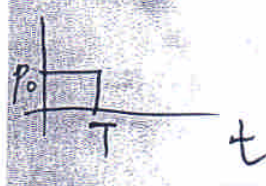


FIG. 2.5. Image displacement and stress pulse system for propagation, interaction, and reflection from a free end.

and (c), there is a doubling effect, but the doubling is associated with the displacement at the end and not the stress. This latter is, of course, always zero. Continuing to step (d), it is seen that the reflected stress pulse is opposite to the incident pulse; thus compression has reflected as tension and vice versa. This stress reversal is a characteristic of the free end.

Consider the applied pressure to be a step pulse described by



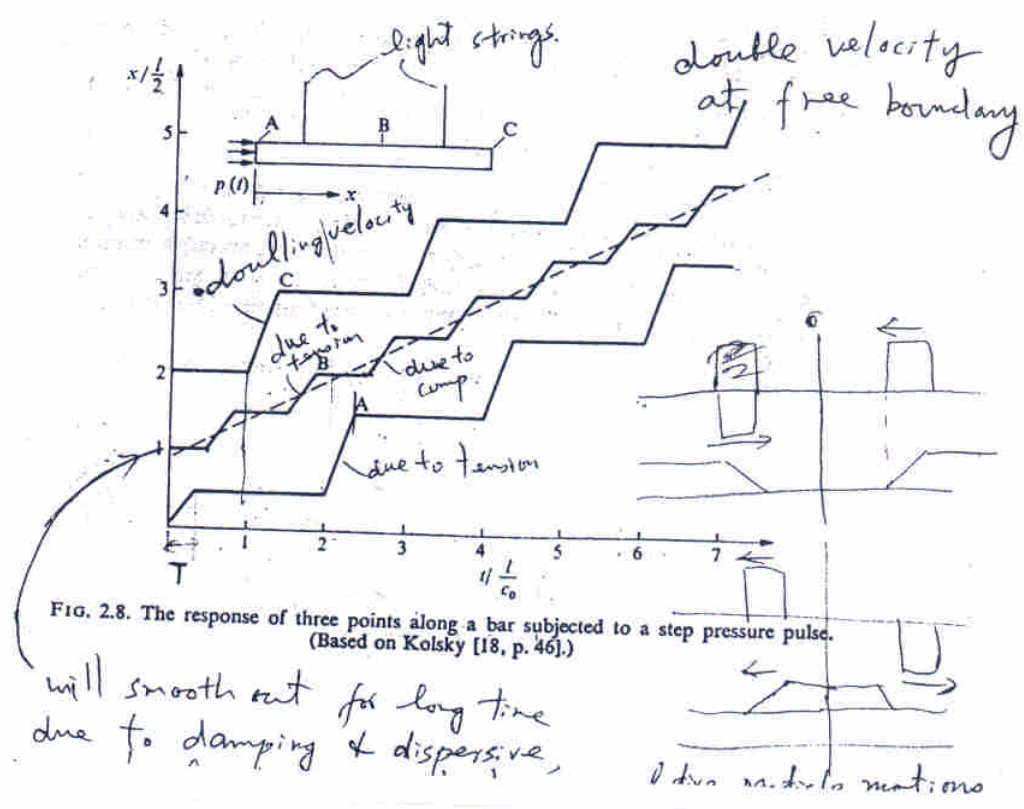
$$p(t) = \begin{cases} -p_0, & 0 < t < T \\ 0, & T < t \end{cases}$$

where the negative sign indicates compression. The propagating stress wave will be given by

$$\sigma(x, t) = p(x - c_0 t), \quad (2.3.2)$$

before the first end reflection. Now the response of a particle at a typical point to this stress pulse can be deduced from the results presented in § 2.1. In particular, eqn (2.1.12) shows that when the step stress pulse reaches $x = x_0$, the particle will be brought from rest to the velocity

$$v(x_0, t) = -c_0 p(x_0 - c_0 t)/E$$

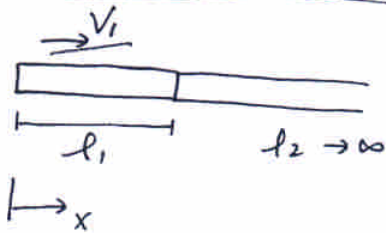


Longitudinal Impact.

9

longitudinal collinear impact of two rods.

consider two similar rods.



$$E_1 = E_2, \quad A_1 = A_2, \quad \rho_1 = \rho_2.$$

$$\underline{V_2 = 0.}$$

when the rods collide at $t=0$, waves propagate to right & left from the point of contact.

$$\begin{cases} u_1(x,t) = f_1(x - c_1 t) + g_1(x + c_1 t) \\ u_2(x,t) = f_2(x - c_2 t) + g_2(x + c_2 t) \end{cases} \quad c_1 = c_2$$

initially, both rods are stress free, and $V_2 = 0$,
velocity of the moving rod = V_1 ,

$$\begin{cases} \underline{N(x,t)} = \frac{\partial u}{\partial t} = -c_1 f'_1(x - c_1 t) + c_1 g'_1(x + c_1 t) \\ \underline{E \frac{\partial u}{\partial x}} = \sigma(x,t) = E \{ f'_1(x - c_1 t) + g'_1(x + c_1 t) \} \end{cases}$$

$\therefore$ the initial condition take the form.

$$\begin{cases} -c_1 f'_1(x) + c_1 g'_1(x) = V_1 \\ EA f'_1(x) + EA g'_1(x) = 0 \\ -c_1 f'_2(x) + c_1 g'_2(x) = 0 \\ EA f'_2(x) + EA g'_2(x) = 0 \end{cases} \quad \text{at } t=0$$

eq. may be solved.

18

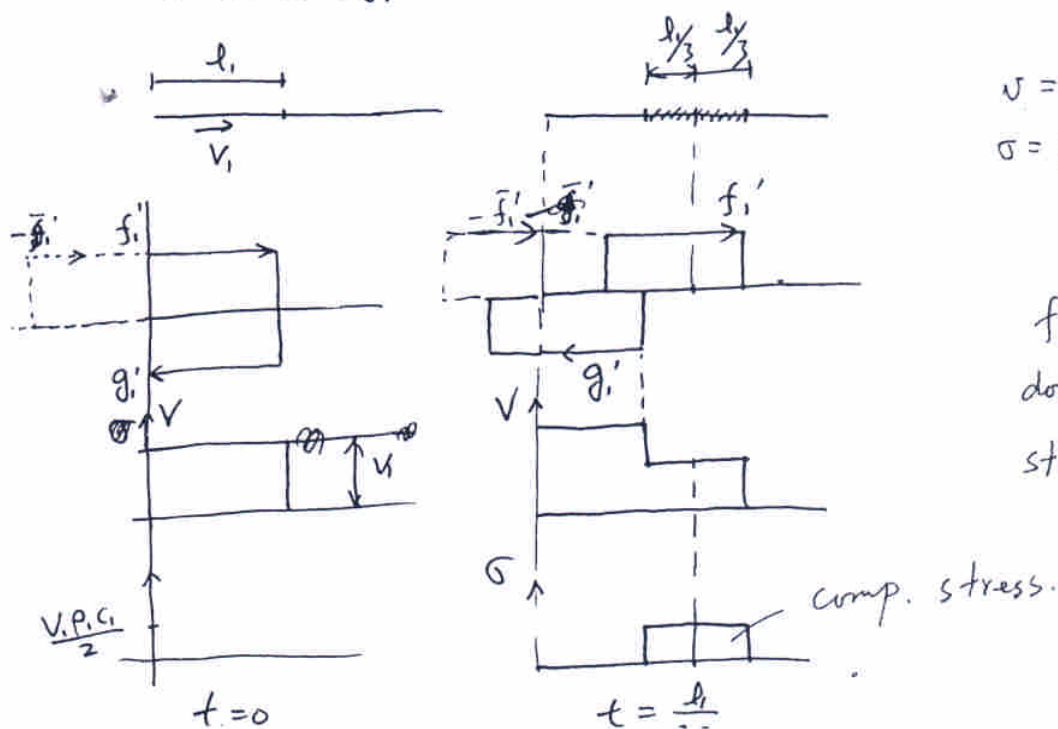
$$\begin{cases} f_2' = g_2' = 0 \\ f_1' = -g_1' = -\frac{V_1}{2c_1} \end{cases} \quad (0 < x < l_1)$$

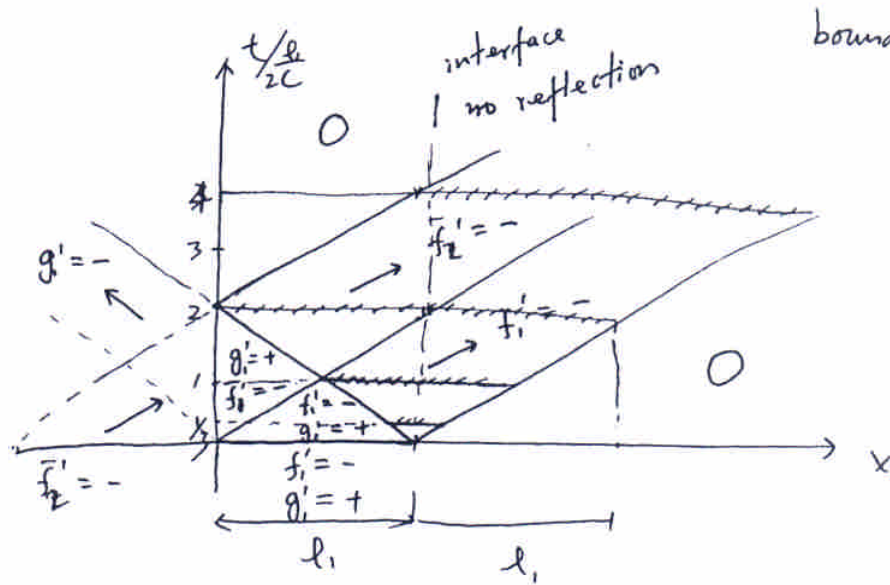
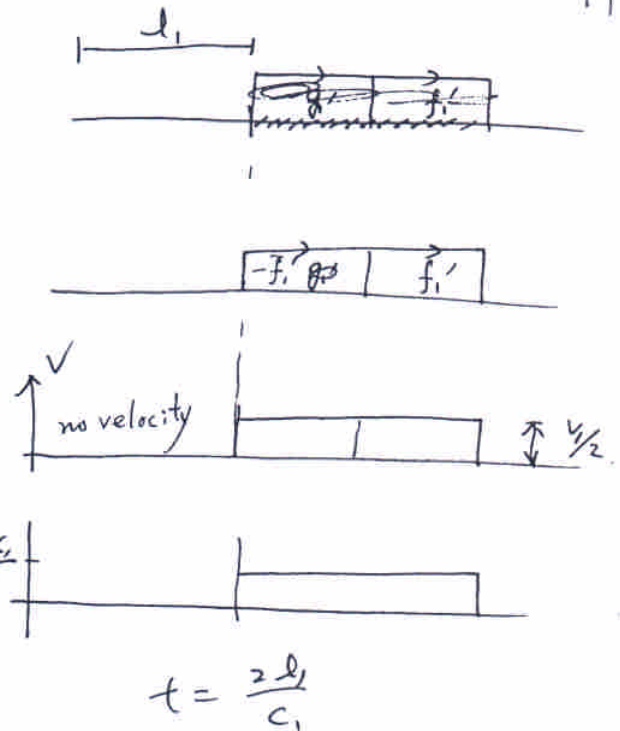
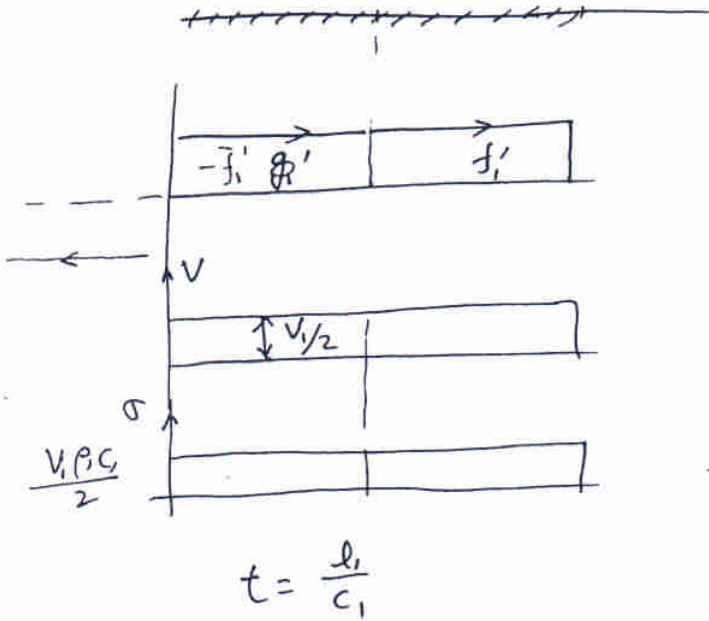
$\therefore$ initially stress free, $\therefore f_1', g_1'$ represented by two rectangular wave shapes

f_1', g_1' have be frozen at $t=0$. but it should be recalled that f_1' is rightward propagation wave
 g_1' " leftward " " "

also. $\therefore$ the rods are identical, $\therefore$ no reflection at the contact surface, i.e. no discontinuity and f_1 moves across $x = l_1$.

the wave g_1' reflects from the free end on the left.
 $[\sigma(0, t) = 0]$.



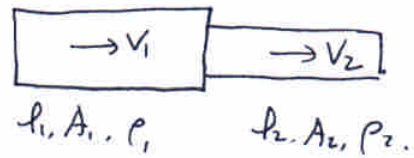


boundary condition.
 $g'_1 \rightarrow -f'_2$

$$V = (-f' + g') \cdot c_1$$

$$\sigma = E(f' + g')$$

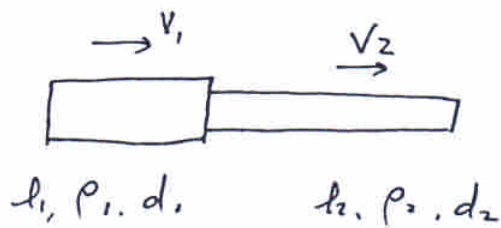
for two different cross-section



$v_1 > v_2$. impact will occur.

transmission and reflection will occur at the interface for each

for the collinear impact. of two rods ^{finite}



same material (steel).

$$l_1 = 18'' \quad l_2 = 2 l_1$$

$$d_1 = 1.5 d_2 \quad d_2 = 1''$$

$$c_1 = c_2 = 2 \times 10^4 \text{ in/s}$$

$$\underline{V_1 = V = 240 \text{ in/s}} \quad \underline{V_2 = 0}$$

at $t = 0$

$$\begin{cases} f_1' = -g_1' = -\frac{V}{2c_1} \\ f_2' = g_2' = 0 \end{cases}$$

for $t > 0$, the transmitted component f_2' and reflected comp. g_2' are

$$\underline{f_2' = \frac{2c_1 E_1 A_1}{A_1 E_1 c_2 + A_2 E_2 c_1} f_1'}$$

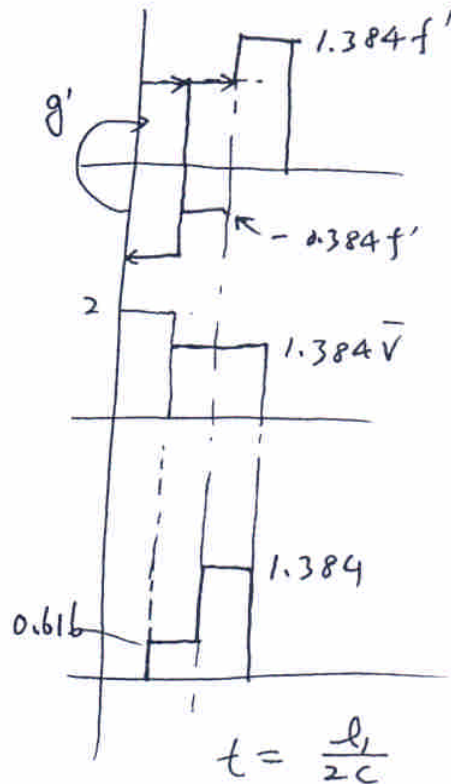
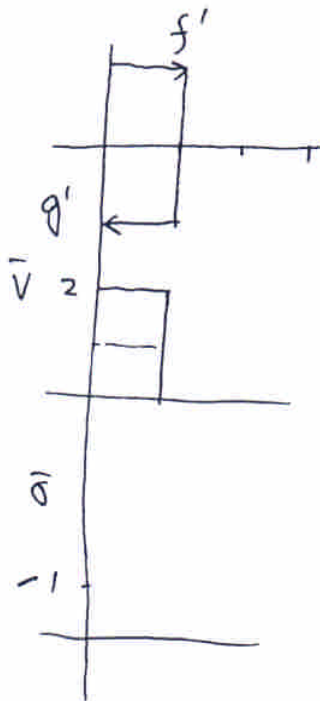
$$\underline{= 1.384 f_1'}$$

$$\underline{g_2' = \frac{c_1 A_2 E_2 - c_2 A_1 E_1}{A_1 E_1 c_2 + A_2 E_2 c_1} f_1'}$$

$$\underline{= -0.384 f_1'}$$

non-dimensionalized all quantities

$$\begin{aligned} \bar{f}', \bar{g}' &= \frac{2c_1}{V} (f', g') & f'_1 = -g'_1 = -\frac{V_1}{2c} \\ \bar{V} &= \frac{2}{V} (f', g') & V = -c_1 \left(\frac{f'_1 - g'_1}{2} \right) \\ \bar{\sigma} &= \frac{2}{\rho_1 c_1 V} (f', g') & \sigma = \frac{V_1 \rho_1 c_1}{2} (f'_1 + g'_1) \end{aligned}$$



$$\bar{V} = -\bar{f}' + \bar{g}'$$

$$\bar{\sigma} = \bar{f}' + \bar{g}'$$

see fig. 2.16 and figure in next page.

2.7.3. The split Hopkinson pressure bar

An experimental apparatus for obtaining dynamic anelastic material properties that utilizes longitudinal elastic wave propagation in rods is called the split Hopkinson pressure bar. Kolsky [19] first proposed this modification of the basic Hopkinson bar. Basically, the method consists of sandwiching

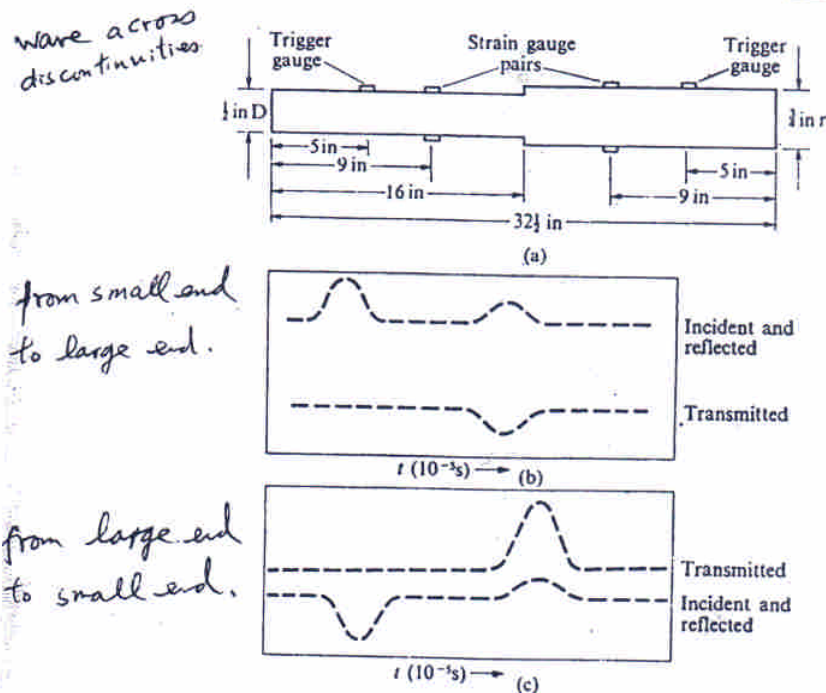


FIG. 2.32. (a) Geometry and arrangement of strain gauges on impact bar for study of longitudinal waves across a discontinuity; (b) experimental results for transmission from the small end toward the large end; (c) results for transmission from the large end to the small end. (After Ripperger and Abramson [35], Figs. 2 and 4)

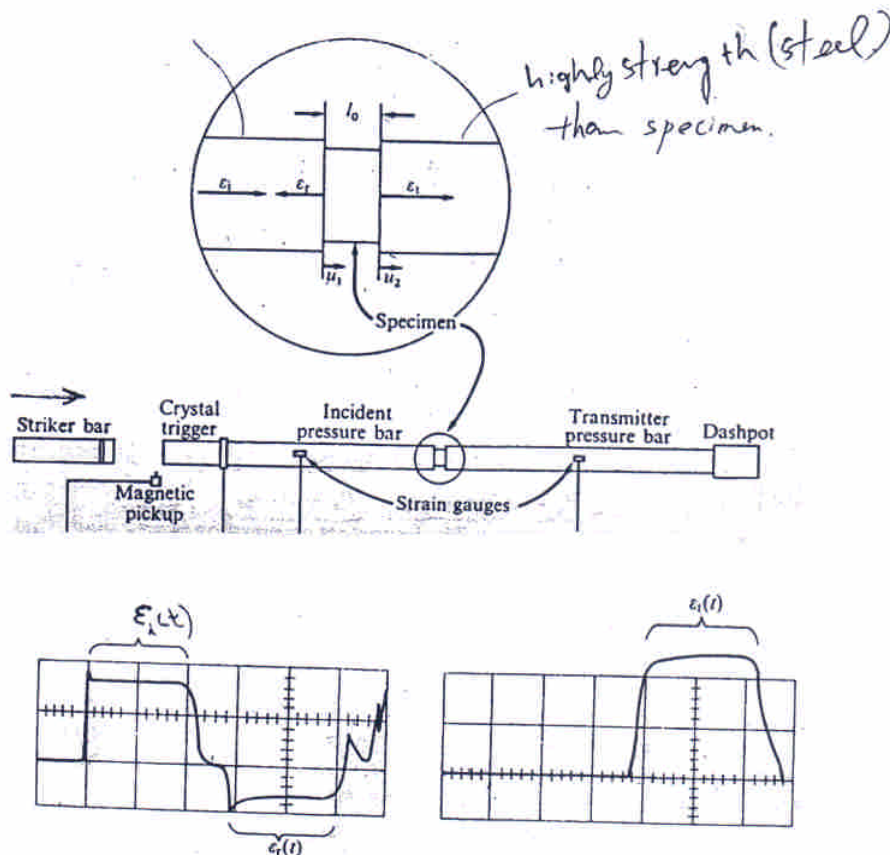


FIG. 2.34. Typical incident, reflected, and transmitted strain signals from a split Hopkinson bar test. (From Lindholm, Yeakley, and Bessey [23], Fig. 2).

Wave in infinite media

16-5-1

chapter 1-4. use strength of material theories
for rods, beams, plates.

assumption : kinematics of deformation.

- kinematics were generally only approximation
of the true deformation.

Improvements in the theories were possible, but
only limited additional information was obtainable.

- In this chapter, exact equation is used
for infinitesimal isotropic elastic media.
- three dimensional problem.
- no boundary interaction (infinite media).

Governing equation.

for homogeneous isotropic elastic solid.

eq. of motion.

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i$$

strain-disp.

relationship.

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

rotation tensor.

$$\omega_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i})$$

generalized Hooke's Law.

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

 λ, μ : Lamé's constants.eq. of motion in terms of disp. may be expressed(Navier's eq.)

$$(\lambda + \mu) u_{j,j i} + \mu u_{i,j j} + \rho f_i = \rho \ddot{u}_i$$

$$\text{dilatation} = \epsilon_{kk} = u_{j,j}$$

by setting body force $f_i = 0$ and setting dilatation $u_{j,j} = 0$ shear stress
state.

$$u_{i,j j} = \frac{\rho}{\mu} \ddot{u}_i$$

$$\text{or } (w_{ik})_{,j j} = \ddot{w}_{ik} / c_s$$

$$c_s = \sqrt{\frac{\mu}{\rho}}$$

speed of propagation of
shear wave.if setting rotation $\omega_{ij} = 0$

$$u_{i,j j} = u_{j,j i}$$

$$\Rightarrow (\lambda + 2\mu) u_{j,j i} = \rho \ddot{u}_i$$

$$\text{or } (u_{i,i})_{,j j} = \rho \ddot{u}_{i,i} / (\lambda + 2\mu) = \frac{\ddot{u}_{i,i}}{c_d}$$

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

speed of propagation of dilatational wave.

18 ~~5-3~~

Helmholtz
Poisson (1828) given a particular form of sol.
may represented by (with $f_i = 0$)

$$u_i = \phi_{,i} + \epsilon_{ijk} \psi_{k,j}$$

$$= \nabla \phi + \nabla \times \vec{\psi}$$

$$\nabla \cdot (\nabla \times \vec{\psi}) =$$

$$\epsilon_{ijk} \psi_{k,ij} = 0$$

skew $\swarrow$ sym $\searrow$ sym. w.r.t.
w.r.t. $i \& j$. $i \& j$.

where $\nabla \cdot \vec{\psi} = 0$ (see back side)

where $\phi(x_i, t)$ and $\vec{\psi}(x_i, t)$ satisfy the wave eq.

$$\phi_{,ii} = \ddot{\phi} / c_d^2$$

$$\psi_{i,jj} = \ddot{\psi}_i / c_s^2$$

$\therefore$ the complete sol of u_i can be expressed as a superposition of a dilatational and as rotation motion with the speeds c_d & c_s .

ϕ & ψ_i are scalar & vector potentials.

also.

$$\frac{c_d}{c_s} = \left(\frac{\lambda + 2\mu}{\mu} \right)^{1/2} = \left(\frac{2 - 2\nu}{1 - 2\nu} \right)^{1/2}$$

$$\therefore 0 \leq \nu \leq 0.5$$

$$\therefore c_d > c_s$$

dilatational waves are called irrotational & primary wave (P wave).

19

rotational waves are called equi-voluminal,
distorsional or secondary waves (S wave)

if $f_i \neq 0$. we also express

$$f_i = c_d^2 \nabla \Phi + c_s^2 \nabla \times \vec{\Psi} = \cancel{f_{,i}} + \cancel{\epsilon_{ijk} B_{kij}} \\ = c_d^2 \Phi_{,i} + c_s^2 \epsilon_{ijk} \Psi_{k,j}$$

governing eq. may decoupled

$$\nabla \{ (\lambda + 2\mu) \nabla^2 \phi + \rho \ddot{\phi} \} + \nabla \times (\mu \nabla^2 \psi_i + \rho \ddot{\psi}_i) = \rho \ddot{f}_i$$

$$\nabla^2 \phi + \Phi = \ddot{\phi} / c_d^2$$

$$\nabla^2 \vec{\psi} + \vec{\Psi} = \ddot{\vec{\psi}} / c_s^2$$

or.

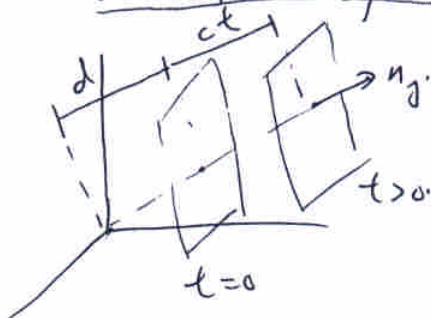
$$\left\{ \begin{array}{l} (\lambda + 2\mu) \nabla^2 \phi + \rho \ddot{\phi} = \rho \ddot{f}_i \\ \mu \nabla^2 \psi_i + \rho \ddot{\psi}_i = \rho \ddot{f}_i \end{array} \right.$$

(see next 4 pages).

c_d, c_s for several materials are given in Table 5.1

Three dimensional plane wave.

plane wave: A solution of the wave eq. represent
 a plane wave, if the solution is constant
over planes which have their normal in the
direction of wave propagation.



n_j : unit outer normal.

consider a plane in the
space x_j , defined by

disp. vector.

$$\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$$

body force

$$\vec{f} = c_d^2 \nabla \Phi + c_s^2 \nabla \times \vec{\Psi}$$

governing eq.

$$\text{or } \begin{cases} (\lambda + \mu) u_{j,j,i} + \mu u_{i,j,j} + \rho f_i = \rho \ddot{u}_i \\ (\lambda + \mu) \nabla(\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} + \rho \vec{f} = \rho \ddot{\vec{u}} \end{cases}$$

$$\begin{aligned} \nabla \cdot \vec{u} &= \nabla \cdot (\nabla \phi + \nabla \times \vec{\psi}) \\ &= \nabla^2 \phi + \nabla \cdot (\nabla \times \vec{\psi}) \rightarrow (\epsilon_{ijk} \psi_{k,j})_{,i} = \epsilon_{ijk} \psi_{k,i,j} = 0 \\ &= \nabla^2 \phi \end{aligned}$$

$$\begin{aligned} \nabla^2 \vec{u} &= \nabla^2 (\nabla \phi + \nabla \times \vec{\psi}) \\ &= \phi_{,i,j,j} + (\epsilon_{ijk} \psi_{k,j})_{,mm} \\ &= (\phi_{,j,j})_{,i} + \epsilon_{ijk} (\psi_{k,mm})_{,j} \\ &= \nabla(\nabla^2 \phi) + \nabla \times (\nabla^2 \vec{\psi}) \end{aligned}$$

$$\begin{aligned} \Rightarrow (\lambda + 2\mu) \nabla(\nabla^2 \phi) + \mu \nabla \times (\nabla^2 \vec{\psi}) + \rho(c_d^2 \nabla \Phi + c_s^2 \nabla \times \vec{\Psi}) \\ = \rho(\nabla \phi + \nabla \times \vec{\psi})'' \end{aligned}$$

$$\begin{cases} \nabla(\nabla^2 \phi + \Phi) = \nabla \ddot{\phi} / c_d^2 & \nabla \times (\nabla^2 \vec{\psi} + \vec{\Psi}) = \nabla \times \ddot{\vec{\psi}} / c_s^2 \\ \text{or } \nabla^2 \phi + \Phi = \ddot{\phi} / c_d^2 & \nabla^2 \vec{\psi} + \vec{\Psi} = \ddot{\vec{\psi}} / c_s^2 \end{cases}$$

$$\underline{x_j \cdot n_j = d} \quad \text{at } t=0.$$

for $t > 0$. assuming the plane moves with speed c in the direction n_j , it propagates a distance ct , $\therefore$ the eq. of the plane is

$$\underline{x_j \cdot n_j = d + ct} \quad \text{or} \quad \underline{x_j \cdot n_j - ct = d}.$$

consider a plane wave

$$\underline{u = \nabla \phi + \nabla \times \vec{\psi}}$$

$$\begin{cases} \phi = \phi(x_j n_j - ct) \\ \phi_{,i} = \phi'(x_j n_j - ct)_{,i} = \phi' \cdot \delta_{ij} n_j \\ \quad = \phi' n_i \\ \phi_{,i\bar{i}} = \phi'' n_i n_i = \phi'' \end{cases} \quad \therefore n_i n_i = 1$$

also. $\underline{\ddot{\phi} = \phi'' \cdot c^2}$

$$\therefore \underline{\phi_{,i\bar{i}} = \ddot{\phi} / c^2}$$

but $\underline{\phi_{,i\bar{i}} = \ddot{\phi} / c_d^2} \quad \therefore \underline{c = c_d}$

$\therefore$ plane waves of ϕ must propagate with speed c_d .

similarly,

$$\underline{\psi_k = \psi_k(x_j n_j - ct)}$$

$$\Rightarrow \underline{\psi_{k, i i} = \ddot{\psi}_k / c^2}$$

$$\therefore \underline{c = c_s.}$$

plane wave of ψ_k must propagate with speed c_s

noting that $\underline{u_i = \phi_{,i} + \epsilon_{ijk} \psi_{k,j}.}$

$\therefore$ plane waves of u_i will have the character of the plane wave of ϕ & ψ_k .

for simplicity, choose direction of propagation along x_1 axis. then $n_1 = 1$, $n_2 = n_3 = 0$, and the form corresponding plane waves ϕ & ψ_k must be.

$$\underline{\phi = \phi(x_1 - c_d t).} \quad \underline{\psi_k = \psi_k(x_1 - c_s t)}$$

and the displacement in x_1 direction is.

$$\begin{aligned} \underline{u_1} &= \phi_{,1} + \epsilon_{ijk} \psi_{k,j} = \phi_{,1} + \psi_{3,2} \oplus \psi_{2,3} \\ &= \underline{\phi'(x_1 - c_d t)} \quad \underbrace{\psi_{3,2} \oplus \psi_{2,3}}_{=0} \end{aligned}$$

$\therefore$ u_1 is the plane dilatational wave with speed c_d . & is directed along the propagation direction, or normal to its plane.

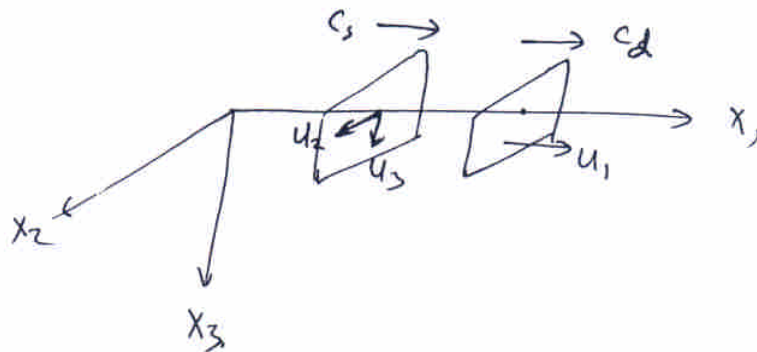
similarly.

235

$$\begin{aligned} u_2 &= \phi_{,2} + \phi_{2,jk} \psi_{k,j} \\ &= \phi_{,2} + \underbrace{\psi_{1,3} - \psi_{3,1}}_{=0} = - \psi_{3,1}'(x_1 - c_s t) \end{aligned}$$

$$u_3 = \phi_{,3} + \psi_{2,1} - \psi_{1,2} = \psi_{2,1}'(x_1 - c_s t).$$

$\therefore$ u_2 & u_3 are a plane equivoluminal wave,
traveling with speed c_s , and is directed
along its plane in the x_2 & x_3 direction
normal to x_1 direction.



or see fig. 5.2.

$\therefore$ u_1, u_2, u_3 is a system of wave that propagate
in the x_1 direction, but are independent
of each other.

if we consider $x_1 x_2$ plane as horizontally
oriented. then

$$\begin{cases} u_1 \text{ wave} \rightarrow \text{P wave.} \\ u_2 \text{ wave} \rightarrow \text{SH wave} \\ u_3 \text{ wave} \rightarrow \text{SV wave.} \end{cases}$$

Plane Harmonic Wave in Elastic Half-Space.

wave reflection, refraction and generation,
 focus on plane wave, harmonic in time & space
 incident P or S wave, after reflection, from boundary P and
 S waves are generated.

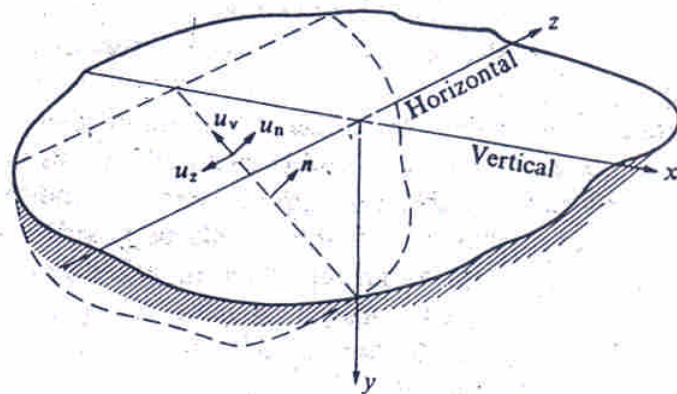


FIG. 6.1. Plane wave, with wave normal $\mathbf{n}$ in the x, y -(vertical) plane advancing toward a free surface.

$$-\infty < x, z < \infty, \quad y \geq 0$$

plane strain in x - y plane

$$u_z = w = \frac{\partial}{\partial z} = 0$$

$$u_i = \nabla \phi + \nabla \times \vec{\psi}$$

for plane strain condition, ϕ & $\vec{\psi}$ can be at
 most function of x & z

$$u_1 = \phi_{,1} + e_{1,jk} \psi_{k,j}$$

$$= \phi_{,1} + \psi_{3,2} - \cancel{\psi_{2,3}}^0$$

$$u_2 = \phi_{,2} + e_{2,jk} \psi_{k,j}$$

$$= \phi_{,2} + \cancel{\psi_{1,3}}^0 - \psi_{3,1}$$

$$\text{or } u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \quad \text{where } \psi = \psi_3$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$$

incident P and SV waves: ϕ_1 & ψ_3

reflected P and SV waves: ϕ_2 & ψ_4

waves are of infinite beam width

the totality of these waves is given by

$$\phi = \phi_1 + \phi_2$$

$$\psi = \psi_3 + \psi_4$$

ϕ & ψ satisfy (without body force)

$$\nabla^2 \phi = \frac{\ddot{\phi}}{c_d^2}$$

$$\nabla^2 \psi = \frac{\ddot{\psi}}{c_s^2}$$

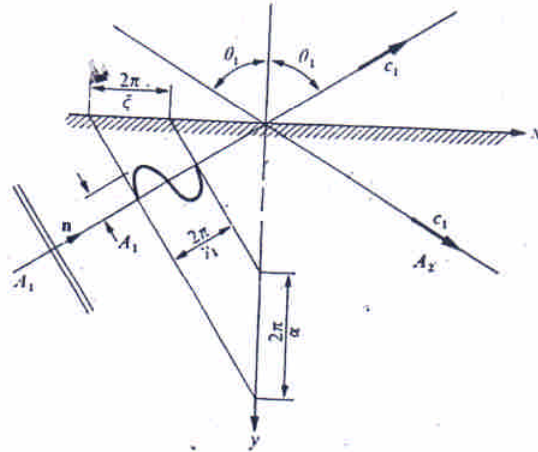


FIG. 6.2. Incident and reflected wave system for $\Phi(x, y, t)$.

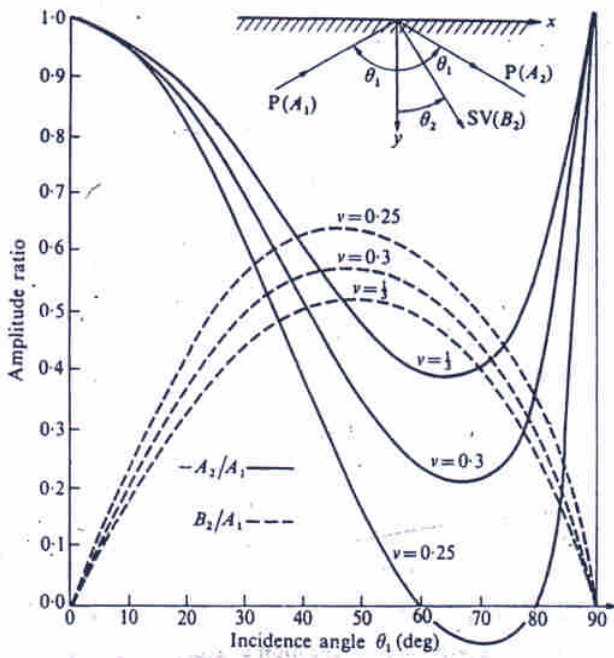


FIG. 6.3. Amplitude ratios $A_2/A_1, B_2/A_1$ for incident P waves, for various Poisson's ratios, with a ray representation of the reflection also shown.

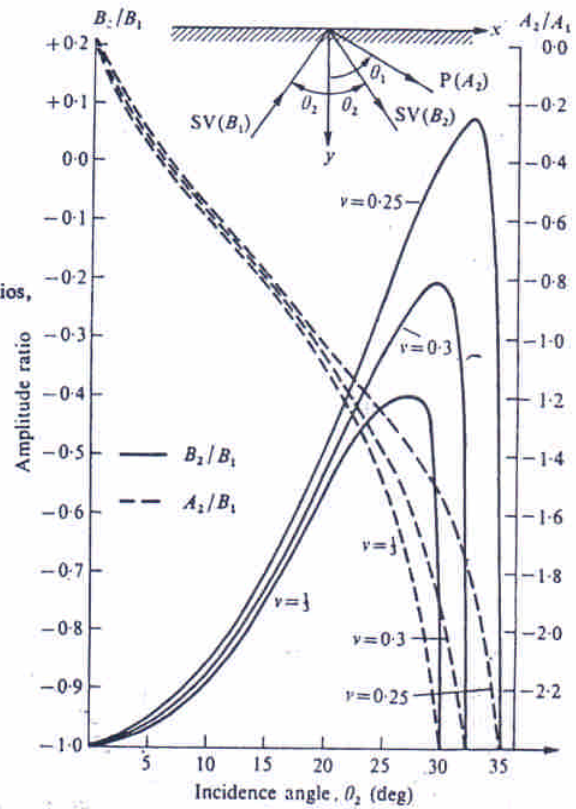


FIG. 6.4. Reflected wave amplitude ratios $A_2/B_1, B_2/B_1$ for incident SV waves and various Poisson's ratios, with the ray representation of the reflection also shown.

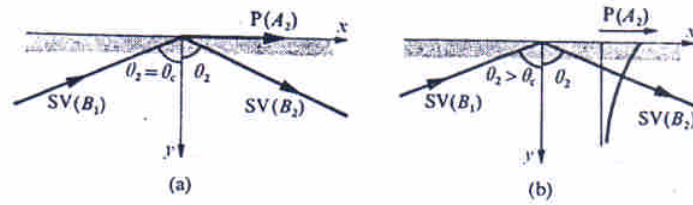


FIG. 6.5. (a) Incident SV waves at the critical angle and (b) greater than the critical angle θ_c .

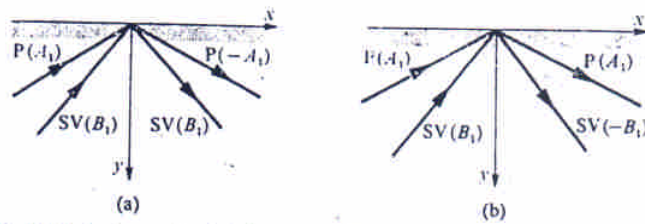


FIG. 6.6. (a) Reflections of pairs of waves, with $A_1 = -A_2$, $B_1 = B_2$, and (b) $A_1 = A_2$, $B_1 = -B_2$.

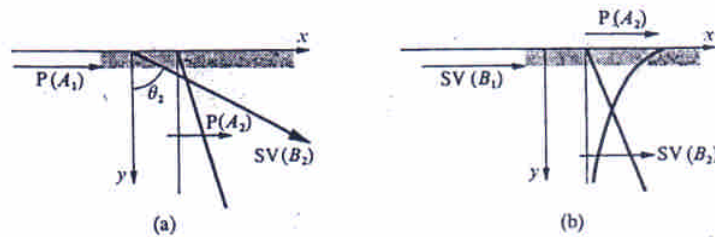


FIG. 6.7. Reflection of waves at grazing incidence with (a) P waves incident and (b) SV waves incident.